

Five Basic Concepts of Axiomatic Rewriting Theory

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Abstract

In this invited talk, I will review five basic concepts of Axiomatic Rewriting Theory, an axiomatic and diagrammatic theory of rewriting started 25 years ago in a LICS paper with Georges Gonthier and Jean-Jacques Lévy, and developed along the subsequent years into a full-fledged 2-dimensional theory of causality and residuation in rewriting. I will give a contemporary view on the theory, informed by my later work on categorical semantics and higher-dimensional algebra, and also indicate a number of current research directions in the field.

A good way to understand Axiomatic Rewriting Theory is to think of it as a 2-dimensional refinement of Abstract Rewriting Theory. Recall that an abstract rewriting system is defined as a set V of vertices (= terms) equipped with a binary relation $\rightarrow \subseteq V \times V$. This abstract formulation is convenient to formulate various notions of termination and of confluence, and to compare them, typically:

strong normalisation	vs.	weak normalisation
confluence	vs.	local confluence

Unfortunately, the theory is not sufficiently informative to capture more sophisticated structures and properties of rewriting systems related to causality and residuation, like

- redexes and residuals
- finite developments
- standardisation
- head rewriting paths

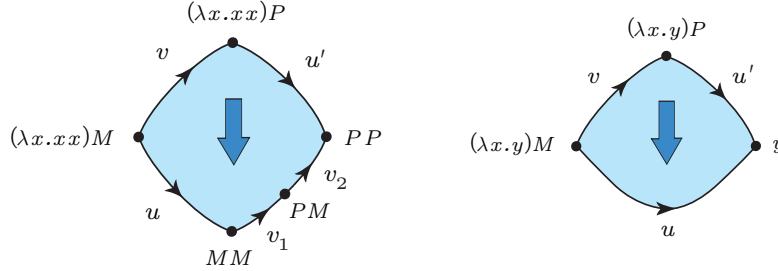
These structures and properties are ubiquitous in rewriting theory. They appear in conflict-free rewriting systems like the λ -calculus as well as in rewriting systems with critical pairs, like action calculi and bigraphs designed by Milner [9] as universal calculus integrating the λ -calculus, Petri nets and process calculi, or the $\lambda\sigma$ -calculus introduced by Abadi, Cardelli, Curien and Lévy [1] to express in a single rewriting system the various evaluation strategies of an environment machine.

It thus makes sense to refine Abstract Rewriting Theory into a more sophisticated framework where the causal structures of computations could be studied for themselves, in a generic way. Intuitively, the causal structure of a rewriting path $f : M \twoheadrightarrow N$ is the cascade of elementary computations implemented by that path. In order to extract these elementary computations from the rewriting path f , one needs to trace operations (= redexes) inside it. This is achieved by permuting the order of execution of independent redexes executed by f . An axiomatic rewriting system is thus defined as a graph $G = (V, E, \partial_0, \partial_1)$ consisting of a set V of vertices (= the terms), a set E of edges (= the redexes) and a pair of source and target functions $\partial_0, \partial_1 : E \rightarrow V$ equipped moreover with a family of permutation tiles, satisfying a number of axiomatic properties.

1. Permutation tiles. The purpose of permutation tiles is to permute the order of execution of redexes. In our axiomatic setting, a permutation tile (f, g) is a pair of coinitial and cofinal rewriting paths of the form:

$$f = M \xrightarrow{v} P \xrightarrow{u'} N \quad g = M \xrightarrow{u} Q \xrightarrow{h} \twoheadrightarrow N$$

where u, v, u' are redexes and h is a rewriting path. The intuition is that h computes the residuals of the redex v along the redex u . Two typical permutation tiles in the λ -calculus are the following one:



where $h = v_1 \cdot v_2$ on the left-hand side and $h = id$ on the right-hand side.

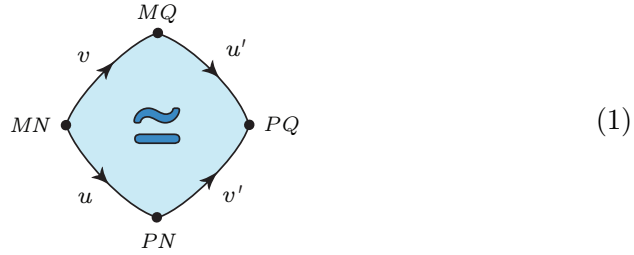
2. Standardisation cells. The permutation tiles are oriented, and generate a 2-dimensional rewriting system on the 1-dimensional rewriting paths. In order to distinguish this rewriting system from the original rewriting system, we call it the standardisation rewriting system. A standardisation path θ between 1-dimensional rewriting paths $f, g : M \twoheadrightarrow N$ is then written as

$$\theta : f \Rightarrow g : M \twoheadrightarrow N$$

The axioms of Axiomatic Rewriting Theory are designed to ensure that this 2-dimensional rewriting system is weakly normalising and confluent. In order to establish weak normalisation, one needs to clarify an important point: when should one consider that two standardisation paths

$$\theta, \theta' : f \Rightarrow g : M \twoheadrightarrow N$$

are equal? The question looks a bit esoteric, but it is in fact fundamental! By way of illustration, consider the following permutation tile in the λ -calculus:



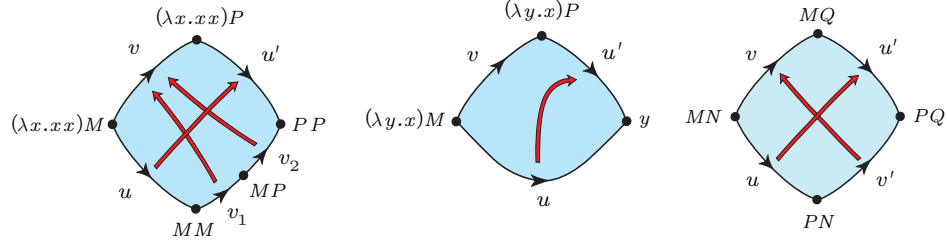
where the two β -redexes u and v should be considered as syntactically disjoint because u is a β -redex of the subterm M and v is a β -redex of the disjoint subterm N . If one does not want to give a left-to-right precedence to the β -redex u over the β -redex v , one should equip the axiomatic rewriting system with two permutation tiles

$$\theta_1 : v \cdot u' \Rightarrow u \cdot v' \qquad \theta_2 : u \cdot v' \Rightarrow v \cdot u'.$$

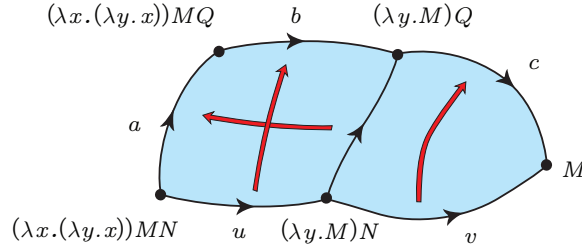
The task of the permutation tile θ_1 is to permute u before v , while the task of the permutation tile θ_2 is to permute v before u . It thus makes sense to require that their composite are equal to the identity in the standardisation rewriting system:

$$\theta_1; \theta_2 = id : v \cdot u' \Rightarrow v \cdot u' \qquad \theta_2; \theta_1 = id : u \cdot v' \Rightarrow u \cdot v'$$

Of course, this enforces that θ_1 and θ_2 are inverse. One declares in that case that the permutation tile (1) is reversible. A standardisation path $\theta : f \Rightarrow g$ consisting only of such reversible permutation tiles is called reversible, and one writes $\theta : f \simeq g$ in that case. A simple and elegant way to describe the equational theory on standardisation paths is to equip every permutation tile (f, g) with an ancestor function $\varphi : [n] \rightarrow [2]$ where $[k] = \{1, \dots, k\}$ and n is the length of the path $g = u \cdot h$. The purpose of the function φ is to map the index of redex in $g = u \cdot h$ to the index of its ancestor $f = v \cdot u'$, in the following way:



By way of illustration, the permutation tiles equipped with their ancestor functions may be composed in the following way in the λ -calculus:



This leads us to identify two standardisation paths $\theta, \theta' : f \Rightarrow g$ when they produce the same ancestor function. A standardisation cell is then defined as an equivalence class of standardisation paths $\theta, \theta' : f \Rightarrow g$ modulo this equivalence relation. Note in particular that the equivalence relation identifies the standardisation path $\theta_1; \theta_2$ with the identity, and similarly for $\theta_2; \theta_1$.

In this way, one defines for every axiomatic rewriting system G a 2-category $\mathbf{Std}(G)$ of whose objects are the vertices (= terms) of G , whose morphisms are the paths (= rewriting paths) of G , and whose 2-cells are the standardisation cells. One declares that two rewriting paths $f, g : M \rightarrow N$ are equivalent modulo redex permutation (noted $f \sim g$) when f and g are in the same connected component of the hom-category $\mathbf{Std}(G)(M, N)$ of rewriting paths from M to N . This means that one can construct a zig-zag of standardisation paths between f and g . We also like to say that the rewriting paths f and g are homotopy equivalent when $f \sim g$.

3. Standard rewriting paths. A rewriting path $f : M \rightarrow N$ is called standard when every standardisation cell $\theta : f \Rightarrow g : M \rightarrow N$ is reversible. The standardisation theorem states that

Standardisation Theorem. For every rewriting path $f : M \rightarrow N$ there exists a standardisation cell $\theta : f \Rightarrow g$ to a standard rewriting path $g : M \rightarrow N$.

Moreover, this standard rewriting path is unique in the sense that for every standardisation cell $\theta' : f \Rightarrow g'$ to a standard rewriting path $g' : M \rightarrow N$, there exists a reversible standardisation path $\theta'' : g' \simeq g$ such that $\theta = \theta'; \theta''$.

The theorem is established in any axiomatic rewriting system G using the elementary axioms on the permutation tiles provided by the theory. As a matter of fact, the property is even stronger: it states that there exists a unique standardisation cell θ from f to the standard rewriting path g . This means that every standard path $g : M \rightarrow N$ is a terminal object in its connected component of rewriting paths $f : M \rightarrow N$. See [3, 4, 8] for details.

4. External rewriting paths. An external rewriting path $e : M \rightarrow N$ is defined as a rewriting path such that for every standard rewriting path $f : N \rightarrow P$, the composite rewriting path $e \cdot f : M \rightarrow P$ is standard. Note in particular that every external rewriting path is standard. Accordingly, a rewriting path $m : M \rightarrow N$ is called internal when for every standardisation cell $\theta : m \Rightarrow e \cdot f$ where the rewriting path e is external, the rewriting path e is in fact the identity on M . One establishes the following property in every axiomatic rewriting system, see [6] for details:

Factorisation Theorem: For every rewriting path $f : M \rightarrow N$, there exists a unique external rewriting path $e : M \rightarrow P$ and a unique internal rewriting path $m : P \rightarrow N$ up to permutation equivalence such that $f \sim e \cdot m$. This factorization is moreover functorial.

5. Head-rewriting paths. The factorization theorem is supported by the intuition that only the external part $e : M \rightarrow P$ of a rewriting path $f : M \rightarrow N$ performs relevant computations, while the internal part $m : P \rightarrow N$ produces essentially useless extra computations. The factorization property plays a fundamental role in the theory. In particular, it enables us to establish a stability theorem which shows the existence of head-rewriting paths in every axiomatic rewriting system, even the rewriting system is non-deterministic and has critical pairs. The stability theorem states that under very general and natural assumptions on a set \mathcal{H} of head-values, see [7], the following property holds:

Stability Theorem: For every term M of the axiomatic rewriting system, there exists a cone of external paths (called head-rewriting paths)

$$e_i \quad : \quad M \rightarrow V_i \quad \quad \text{with } V_i \in \mathcal{H}$$

indexed by $i \in I$, which satisfies the following universality property: for every rewriting path $f : M \rightarrow W$ reaching a head-value $W \in \mathcal{H}$, there exists a unique index $i \in I$ such that the rewriting path f factors as

$$f \sim e_i \cdot h \quad : \quad M \rightarrow W$$

for a given rewriting path $h : V_i \rightarrow W$. The rewriting path $h : V_i \rightarrow W$ is moreover unique modulo permutation equivalence. In the case of axiomatic rewriting systems without critical pairs, the theorem establishes the existence of a head-rewriting path $e : M \rightarrow V$ for every term M which can be rewritten to a head-value $W \in \mathcal{H}$. The stability theorem is particularly useful in rewriting systems with critical pairs. By way of illustration, it enables one to describe the head-rewriting paths $e_i : M \rightarrow V_i$ which transport a λ -term M to its head-normal forms in the $\lambda\sigma$ -calculus, see [5] for details.

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